Final Exam Dynamical Systems I

21-11-2001: 13.00h - 16.00h

1. The map $f = f_n : \mathbb{S}^1 \to \mathbb{S}^1$ is defined by

$$f(\theta) = \theta + \frac{2\pi}{n} + \frac{1}{n+1} \sin n\theta,$$

where $n \in \mathbb{Z}_{>0}$.

   (a) Sketch the phase portrait of f_3.

   (b) Prove that f_n is a diffeomorphism.

   (c) Show that f has exactly two periodic orbits of minimal period n. Determine the stability types of these orbits.

   (d) Compute the rotation number of f.

   (e) Can f have periodic points of other minimal period than n ? Explain your answer.

   (f) Is f Morse-Smale? Explain your answer.

   (g) Give the definition of a chaotic evolution. Does f have chaotic evolutions? Explain your answer.

   (h) Give the definition of sensitive dependence on initial conditions. Are there points where f has sensitive dependence on initial conditions? Explain your answer.

2. Consider the vector field

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 - \mu, \end{aligned}$$

with $(x, y) \in \mathbb{R}^2$, and parameter $\mu \in \mathbb{R}$.

   (a) Compute the equilibria of this system and their stability types, depending on μ.

   (b) Determine the bifurcation points (of equilibria), and draw a bifurcation diagram.

   (c) For each bifurcation point, sketch a phase portrait for all qualitatively different cases, that is, for μ to the left and right of the bifurcation point, and at the bifurcation point.

   (d) In these phase portraits, indicate the stable and unstable manifolds of saddle points.

3. Let $f = f_\alpha : [0, 1] \longrightarrow \mathbb{R}$ be the map defined by

$$f_\alpha(x) = \alpha(2x \text{ mod } 1),$$

where α is a parameter. Throughout this exercise we take $\alpha > 1$.

(a) Sketch the map f for $\alpha = 2$.

(b) Let $\Lambda \subset [0, 1]$ be the set of all points whose forward orbit is contained in $[0, 1]$. Explain how to construct Λ.

(c) Prove that Λ is a nonempty compact set of zero Lebesgue measure.

(d) Compute the (largest) Lyapunov exponent at any $x \in \Lambda$.

(e) Give the definition of an eventually periodic point (in the case of maps), and show that the set of eventually fixed points of f is contained and dense in Λ.
Hint: use symbolic dynamics. Describe the symbol set, a suitable map on this set, and a conjugacy between this map and f. You do not need to show that the conjugacy is a homeomorphism, but you have to prove that it conjugates the two maps.

(f) Prove that Λ is a Cantor set.

(g) Determine the box counting dimension of Λ.

(h) **Bonus exercise for an extra half point**: Compute the topological entropy of $f|_\Lambda$.